

CONFORMAL SPECTRUM AND HARMONIC MAPS

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ABSTRACT. This paper is devoted to the study of the conformal spectrum (and more precisely the first eigenvalue) of the Laplace-Beltrami operator on a smooth connected compact Riemannian surface without boundary, endowed with a conformal class. We give a constructive proof of a critical metric which is smooth except at some conical singularities and maximizes the first eigenvalue in the conformal class of the background metric. We also prove that the map associating a finite number of eigenvectors of the maximizing λ_1 into the sphere is harmonic, establishing a link between conformal spectrum and harmonic maps.

CONTENTS

1. Introduction	1
2. Notations and results	4
3. Construction of an extremalizing sequence of metrics	7
3.1. Measure estimates	8
3.2. Control of the eigenfunctions	13
4. Regularity <i>a priori</i> of limiting densities	15
5. Proofs of Theorems 2.1 and 2.2	20
References	24

1. INTRODUCTION

Let (M, g) be a smooth connected compact Riemannian surface without boundary. In this paper, we construct a map from the manifold M into the sphere by means of eigenvectors of the first eigenvalue of the Laplace-Beltrami on (M, \tilde{g}) where \tilde{g} is conformal to g and maximizes the first eigenvalue of the Laplace-Beltrami operator. More precisely, denote by $A_g(M)$ the area of the surface (M, g) and denote Δ_g the Laplace-Beltrami operator on (M, g) . The spectrum of $-\Delta_g$ consists in the sequence $\{\lambda_k(g)\}_{k \geq 0}$ and satisfies

$$\lambda_0(g) = 0 < \lambda_1(g) \leq \lambda_2(g) \leq \dots \leq \lambda_k(g) \leq \dots$$

If we assume that the area $A_g(M)$ is normalized by one then by the fundamental result of Korevaar (see [Kor93] and also [YY80]), it follows that every $\lambda_k(g)$ for a given $k \geq 0$ has a universal bound depending on the topological type of M over all the metrics g with normalized area.

More precisely, denote

$$\Lambda(M) = \sup_g \lambda_1(g) A_g(M)$$

where the supremum is taken over all smooth Riemannian metrics g on the manifold M . It is a well-known result that $\Lambda(M) < \infty$ and it has been proved in [YY80] that for an orientable surface of genus γ , we have (see also [Kor93])

$$\Lambda(M) \leq 8\pi(\gamma + 1).$$

This allows to define a topological spectrum on M for $-\Delta_g$ by taking upper bounds of the eigenvalues λ_1 .

In the last years, several works have been devoted to explicit computations of the quantity $\Lambda(M)$. For a surface of genus zero, Hersch (see [Her70]) proved that

$$\Lambda(\mathbb{S}^2) = 8\pi.$$

In the case of non-orientable surfaces, Li and Yau [LY82] proved the following equality

$$\Lambda(\mathbb{R}P^2) = 12\pi$$

and as far as the quantity $\Lambda(M)$ is concerned, one of the author (see [Nad96]) proved that

$$\Lambda(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}.$$

A result of Yang and Yau [YY80] ensures that

$$\Lambda(M) \leq 8\pi \left[\frac{\gamma + 3}{2} \right]$$

for any surface of arbitrary genus γ and $[\cdot]$ in the right hand side stands for the integer part. As far as the Klein bottle is concerned, we refer the reader to [JNP06] and [ESGJ06].

The above discussion gives rise to two related problems: to obtain precise upper bound for $\Lambda(M)$ depending on the genus of the surface; to obtain a sharp bound for λ_1 in a given conformal class of the surface. Obviously any progress on each of these two problems gives information on the other one.

Before dwelling much into topological spectrum, we define

Definition 1.1. *A smooth connected compact Riemannian manifold (M, g) is called a λ_1 -maximal manifold if the metric g realizes the supremum in $\Lambda(M)$.*

Remark 1.2. *Note that, following [ESI00a], an extremal metric for the first eigenvalue is a critical point g_0 of the functional λ_1 , i.e. for any analytic deformation g_t of the Riemannian metric g_0 in the class of metrics of fixed volume, we have*

$$\lambda_1(g_t) \leq \lambda_1(g_0) + o(t), \quad t \rightarrow 0$$

For instance, in Hersch's result (see [Her70]), \mathbb{S}^2 endowed with a round metric is actually λ_1 -maximal. Similarly, $\mathbb{R}P^2$ with its standard metric is λ_1 -maximal (see [LY82]) and the flat equilateral torus is the only λ_1 -maximal torus (see [Nad96]). This latter fact induces some consequences on the Berger's isoperimetric problem (see [Ber73, Nad96]). On the other hand, an isometric immersion φ from (M, g) in the sphere is a minimal immersion if and only if it satisfies

$$-\Delta_g \varphi = \lambda \varphi.$$

If λ is the first eigenvalue of the laplacian then the manifold (M, g) is said to be λ_1 -minimal. For instance, any Riemannian irreducible homogeneous space is λ_1 -minimal. In [Nad96], the first author proved the following result: any λ_1 -maximal Riemannian surface is λ_1 -minimal. This result has been generalized by El Soufi and Ilias to any dimension in [ESI00b]. The importance of maximal metrics in Riemannian geometry is related to λ_1 -minimality. The metric g on an n -dimensional manifold is λ_1 -minimal if the eigenspace $U_1(g)$ associated to the first non zero eigenvalue of the Laplace-Beltrami operator contains a family $\{u_1, \dots, u_k\}$ of eigenfunctions such that

$$(1) \quad g = \sum_{i=1}^k du_i \otimes du_i.$$

It appears that the topological spectrum has deep connections with minimal submanifolds of Euclidean spheres. Indeed, by a well-known result of Takahashi (see [Tak66]), there is equivalence between the two assertions: the map

$$U = (u_1, \dots, u_k)$$

is a minimal immersion from (M, g) into the Euclidean sphere \mathbb{S}_1^{k-1} if and only if the metric g writes as (1).

The hardest question on the existence of a smooth, or at least sufficiently smooth, metric maximizing the first eigenvalue reminded open.

In a natural ramification of this problem, one can consider a topological spectrum under additional constraints of staying in the conformal class of the background metric. This leads to the so-called conformal spectrum. We define

$$\tilde{\Lambda}(M, [g]) = \sup_{\tilde{g} \in [g], A_{\tilde{g}}(M)=1} \lambda_1(\tilde{g})$$

where $[g]$ is the conformal class of g . Recently, a lot of attention has been devoted to the conformal spectrum on surfaces. For instance, isoperimetric inequalities have been obtained in [CES03, ESIR99] in a conformal class context. Li and Yau (see [LY82]) also discovered a bound between the conformal spectrum (the first eigenvalue) and the conformal volume. The following important inequality was proved in [CES03],

$$\tilde{\Lambda} \geq 8\pi.$$

The central purpose of the present paper is to establish a link between the conformal spectrum and the harmonic maps of the surface into the Euclidean spheres. We prove the existence of an extremalizing metric for $\tilde{\Lambda}(M)$ and provide its regularity.

Our construction is rather explicit in the sense that it is based on an approximation procedure. We prove that there exists a smooth, up to a finite discrete set of points on M , metric in the conformal class $g' \in [g]$ such that it maximizes $\lambda_1(g')$. This provides in a two-dimensional framework a quite complete picture by considering the map generated by several eigenfunctions of the extremalizing metric.

We would like also to mention a recent preprint by Kokarev [?] devoted to similar problems. The results of Kokarev are somehow complementary of ours, though there is no direct overlapping.

2. NOTATIONS AND RESULTS

Let (M, g) be a two-dimensional Riemannian manifold. In local coordinates (x_i, y_i) , the metric writes $g = \sum g_{ij} dx_i dy_j$ and the Laplace-Beltrami operator has the form

$$\Delta_g = \frac{1}{|g|} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial y^j} \right)$$

where we have used the usual convention of repeated indexes and $g^{ij} = (g_{ij})^{-1}$, $|g| = \det(g_{ij})$. We now drop the notation $A_g(M)$ to call it A_g .

We denote by $\lambda_1(g)$ the first non-zero eigenvalue of Δ_g and we have

$$\lambda_1(g) = \inf_{u \in E} R_{M,g}(u)$$

where $R_{M,g}(u)$ is the so-called Rayleigh quotient given by

$$R_{M,g}(u) = \frac{\int_M |\nabla u|^2 dA_g}{\int_M u^2 dA_g}$$

and the infimum is taken over the space

$$E = \left\{ u \in H^1(M), \int_M u = 0 \right\}.$$

Due to the scaling property of the first eigenvalue under a metric change cg , it is natural to introduce a normalization for the metric and we denote by $\mathcal{A}(g)$ the set of all metrics on M satisfying $A_g(M) = 1$. We then consider on M the class $[g]$ of metrics conformal to g in $\mathcal{A}(g)$, i.e.

$$[g] = \{ g' \in \mathcal{A}(g), g' = \mu g, \mu > 0, \mu : M \rightarrow \mathbb{R}, \mu \in L^1(M) \}.$$

Remark 2.1. Notice that in the previous definition, we do not make any a priori assumption on the regularity of the map μ , except of its summability.

In dimension 2, the Laplace-Beltrami operator is conformally covariant in the following sense: if $g' \in [g]$ and $g' = \mu g$, we have

$$(-\Delta_{g'}) = \frac{1}{\mu}(-\Delta_g)$$

and the surface element is conformally changed by the law

$$dA_{g'} = \mu dA_g.$$

We are interested in studying the analogous of the quantity $\Lambda(M)$ previously defined in the context of conformal metrics, i.e.

$$\tilde{\Lambda}(M, [g]) = \sup_{g' \in [g]} \lambda_1(g').$$

We state now our results. We first prove an existence and regularity result on the maximizing metric.

Theorem 2.1. *Let (M, g) be a smooth connected compact boundaryless Riemannian surface. Assume that $\tilde{\Lambda}(M) > 8\pi$. Then there exists a metric $\bar{g} \in [g]$, $\bar{g} = \mu g$, where μ is a smooth function positive outside a finite number of points, such that the metric \bar{g} extremalizes the first eigenvalue in the conformal class of g , i.e.*

$$(2) \quad \lambda_1(\bar{g}) = \sup_{g' \in [g]} \lambda_1(g').$$

Theorem 2.1 implies the following characterization of the metric.

Theorem 2.2. *Let (M, \bar{g}) be the Riemannian manifold endowed with the maximizing metric \bar{g} . Denote $U_1(\bar{g})$ the eigenspace associated to $\lambda_1(\bar{g})$. Then there exists a family of eigenvectors $\{u_1, \dots, u_\ell\} \subset U_1(\bar{g})$ such that the map*

$$(3) \quad \begin{cases} \phi : M \rightarrow \mathbb{R}^\ell \\ x \rightarrow (u_1, \dots, u_\ell) \end{cases}$$

is a harmonic map into the sphere $\mathbb{S}^{\ell-1}$.

Before quoting a corollary of the previous theorem, we define

Definition 2.2. *A conical singularity at $p \in M$ is the following: there exists a neighborhood \mathcal{U} of p in M such that \mathcal{U} is asymptotically diffeomorphic to a cone.*

The previous theorem admits the following corollary.

Corollary 2.3. *Let (M, g) be a smooth connected compact boundaryless Riemannian surface. Assume that $\tilde{\Lambda}(M, [g]) = \Lambda(M) > 8\pi$. Then there exists a metric \bar{g} smooth outside a finite number of conical singularities (i.e. there exists $K \in \mathbb{N}$ and $\{p_k\}_{k=1, \dots, K} \in M^K$ such that $\mu \in C^\infty(M)$ and $\mu > 0$ on $M \setminus \{p_1, \dots, p_K\}$) such that the map*

$$(4) \quad \begin{cases} \phi : M \rightarrow \mathbb{R}^\ell \\ x \rightarrow (u_1, \dots, u_\ell) \end{cases}$$

is a minimal conformal immersion into the sphere $\mathbb{S}^{\ell-1}$.

Proof. By Theorem 2.1 there exists a metric $\bar{g} \in [g]$, such that $\lambda_1(\bar{g}) = \tilde{\Lambda}(M, [g])$. Since by our assumption $\tilde{\Lambda}(M, [g]) = \Lambda(M)$ the metric \bar{g} extremalized λ_1 also with respect to variations of the conformal class of the metric. Hence by the result of [Nad96] the corollary follows (see also [CES03]). □

The proofs of the previous theorems rely on a careful analysis of a Schrödinger type operator. Indeed consider $g' \in [g]$, by conformal covariance, the equation $-\Delta_{g'} u = \lambda_1(g') u$ reduces to the following system

$$(5) \quad \begin{cases} -\Delta_g u = \lambda_1(g') \mu u, & \text{on } M \\ \int_M \mu dA_g = 1. \end{cases}$$

We cannot assume from the beginning that the extremalizing metric μg belongs to the smooth category but instead we will prove that this is the case up to a finite number of conical singularities. The strategy of the proof is the following:

- (1) We first regularize the problem by considering an extremalizing sequence of densities $\{\mu_N\}_N$ in a space of probability measures with bounded densities and of indefinite sign.
- (2) We then prove *a priori* regularity results on the extremal metric.
- (3) We then pass to the limit.

As previously mentioned, the proof of Theorems 2.1 and 2.2 go by an approximation procedure together with careful estimates on the "bad" sets where the density μ might have some inappropriate behaviour.

3. CONSTRUCTION OF AN EXTREMALIZING SEQUENCE OF METRICS

This section is devoted to the construction of an extremalizing sequence of metrics for problem (5). To do so, we perform a regularization by considering it as limit of bounded densities. More precisely, denote by S_N the class of densities μ such that $-\frac{1}{2} \leq \mu \leq N$, $\int_M \mu dA_g = 1$ and $\lambda_1(\mu g)$ satisfies The Schrödinger equation in (5). Introduce the following quantity for $N > 0$

$$\tilde{\Lambda}_N = \sup_{\mu \in S_N} \lambda_1(\mu).$$

The following result is standard (see [LY82, YY80]) and relies on a compactness argument. We denote $\mathcal{M}(M)$ the set of Radon measures on the manifold M .

Proposition 3.1. *For any given $N > 0$, there exists a sequence $\{\mu_{k,N}\}_{k \geq 0}$ such that*

$$\mu_{k,N} \rightharpoonup^* \mu_N \text{ weakly in } \mathcal{M}(M)$$

and

$$\lambda_1(\mu_{k,N} g) \rightarrow \tilde{\Lambda}_N.$$

Furthermore, we have

$$\int_M \mu_N dA_g = 1$$

and

$$-\frac{1}{2} \leq \mu_N \leq N.$$

Of course, the previous proposition relies on the universal bounds for the first non zero eigenvalue in conformal classes discovered by Li and Yau (see [LY82]). The whole point by now is to pass to the limit $N \rightarrow +\infty$ and to prove that the limit obtained this way is indeed a nonnegative density, with sufficient regularity. This amounts to control the two following subsets of M

$$E_-^N = \left\{ x \in M, \quad -\frac{1}{2} \leq \mu_N(x) \leq 0 \right\}$$

and

$$E_N = \{x \in M, \quad \mu_N(x) = N\}.$$

3.1. Measure estimates. We have first the following easy lemma.

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$A_g(E_N) \leq C/N.$$

Proof. The density $\mu_N g$ is of area one, i.e.

$$\int_M \mu_N dA_g = 1.$$

Writing

$$\int_M \mu_N dA_g = \int_{E_N} \mu_N dA_g + \int_{M \setminus E_N} \mu_N dA_g,$$

leads

$$\int_M \mu_N dA_g = NA_g(E_N) + \int_{M \setminus E_N} \mu_N dA_g.$$

We then have since $\mu_N \geq -1/2$

$$\int_{M \setminus E_N} \mu_N dA_g > -\frac{1}{2}A_g(M \setminus E_N).$$

Writing $A_g(M \setminus E_N) = A_g(M) - A_g(E_N)$ gives the desired result. \square

We now come to the measure estimate of the set E_-^N . We have the following general lemma in the plane.

Lemma 3.3. *For any positive constant N there exists an $\epsilon = \epsilon(N) > 0$ such that if $E \subset B(0, 1) \subset \mathbb{R}^2$ is a measurable set,*

$$|E| < \epsilon$$

and $v > 0$ in $B(0, 1)$ is a solution in $B(0, 1)$ of the following differential inequality

$$(6) \quad -\Delta v - hv \leq 0,$$

where h satisfies the inequalities $h < N$ on E , $h < -1/N$ on $B(0, 1) \setminus E$. Then we have

$$v(0) < \frac{1}{2\pi} \int_{S(0,1)} v ds$$

Proof. The following Harnack inequality (see [?]) and bounds for the ground state of Shrödinger operators are well known (see [LL01], Th. 12.4).

Lemma 3.4. *Let $v > 0$ be a solution in $B(0, 2)$ of the Schrödinger equation*

$$-\Delta v + Vv = 0,$$

where $|V| < N$. Then

$$\sup_{B(0,1)} v / \inf_{B(0,1)} v < C,$$

where $C = C(N) > 0$.

Lemma 3.5. *Let v be a solution of the Dirichlet problem*

$$(7) \quad \begin{cases} -\Delta v - Vv = 0, & \text{in } B(0, 1) \\ v = 0, & \text{on } S(0, 1) \end{cases}$$

Then for any $p > 1$ there is a constant $c(p) > 0$ such that if

$$\|V^+\|_p < c(p)$$

then $v \equiv 0$ where V^+ is the positive part of the potential V .

By the previous lemma, we deduce that the eigenvalue of (7) is negative. Let v be a solution of inequality (6). Consider the Dirichlet problem

$$(8) \quad \begin{cases} -\Delta u - hu = 0 & \text{in } B(0, 1), \\ u = v & \text{on } S(0, 1), \end{cases}$$

where $h \in L^\infty(M)$ satisfies the inequalities of Lemma 3.3. By Lemma 3.5, it follows that for sufficiently small $\epsilon > 0$ the Dirichlet problem (8) has a unique solution $u > 0$ in $B(0, 1)$.

We introduce the Green function (with pole at 0) in $B(0, 1)$

$$(9) \quad \begin{cases} -\Delta G + VG = \delta_0 & \text{in } \mathcal{D}'(B(0, 1)) \\ G = 0 & \text{on } S(0, 1). \end{cases}$$

where $V = N$ on $B(0, 1) \setminus B(0, 1 - \epsilon)$ and $V = 0$ on $B(0, 1 - \epsilon)$. The function G is radially symmetric. It follows from the Fredholm alternative that for sufficiently small $\epsilon > 0$ such a Green function exists. Then for any $\delta > 0$ there is an $\epsilon > 0$, $\epsilon = \epsilon(\delta, N)$, such that

$$\int_{S(0, 1-\epsilon)} \frac{\partial G}{\partial r} ds < (1 - \delta) \int_{S(0, 1)} \frac{\partial G}{\partial r} ds.$$

Let $w > 0$ be a solution of the Dirichlet problem

$$(10) \quad \begin{cases} -\Delta w + Nw = 0 & \text{in } B(0, 1) \setminus B(0, 1 - \epsilon), \\ -\Delta w - w/N = 0 & \text{in } B(0, 1 - \epsilon), \\ w = v & \text{on } S(0, 1). \end{cases}$$

There exists $\delta > 0$ such that for sufficiently small $\epsilon > 0$ we will have the inequality

$$w(0) < (1 - \delta) \int_{B(0, 2)} w ds.$$

By Lemma 3.4, for any $\delta > 0$ there is a constant $C = C(N, \delta) > 0$ such that

$$\sup_{B(0, 1-\delta)} u < C \int_{B(0, 1-\delta)} u ds.$$

Set $q = u - w$, $E' = E \cap B(0, 1 - \delta)$. Then

$$(11) \quad \begin{cases} -\Delta q < 0 & \text{in } B(0, 1) \setminus E', \\ -\Delta q < Nu & \text{in } E', \\ q = 0 & \text{on } S(0, 1), \end{cases}$$

Thus we have

$$q(0) < C\epsilon \int_{B(0, 1-\delta)} u ds.$$

hence for sufficiently small $\epsilon > 0$

$$u(0) < (1 - \delta) \int_{B(0, 1)} u ds,$$

$\delta > 0$. From the last inequality immediately follows that

$$v(0) < (1 - \delta) \int_{B(0, 1)} v ds.$$

Lemma 3.3 is proved. □

As an immediate corollary of Lemma 3.3 we have

Lemma 3.6. *For any positive constant N there exists an $\epsilon = \epsilon(N) > 0$ such that if $E \subset B(0, 2) \subset \mathbb{R}^2$ is a measurable set,*

$$|E| < \epsilon$$

and $v > 0$ in $B(0, 2)$ is a solution in $B(0, 2)$ of the following differential inequality

$$(12) \quad -\Delta v - hv \leq 0,$$

where h satisfies the inequalities $h < N$ on E , $h < -1/N$ on $B(0, 2) \setminus E$, then

$$v(0) < \frac{1}{3\pi} \int_{B(0,2) \setminus B(0,1)} v dx$$

Considering a local conformal structure on M we can lift the last lemma on M :

Lemma 3.7. *There exists r_0 such that for each $x \in M$ and $0 < r < r_0$, there is an open set G in a geodesic disk $B(x, r)$ of radius r centered at x such that there exists a positive function $q \in C(B(x, r))$, $q > 0$, such that for any positive constant N there exists an $\epsilon = \epsilon(N) > 0$ such that $E \subset B(x, r)$ being a measurable set,*

$$A_g(E) < \epsilon$$

and $v > 0$ in $B(x, r)$ be a solution in $B(x, r)$ of the following differential inequality

$$(13) \quad -\Delta v - hv \leq 0,$$

where h satisfies the inequalities $h < N$ on E , $h < -1/N$ on $B(x, r) \setminus E$, then

$$v(x) < \int_G q v dA_g / \int_G q dA_g$$

Remark 3.8. *As a function q one can take the jacobian of a conformal map of $B(x, r)$ on the unit disk on the plane.*

Lemma 3.9. *Let \hat{E} be the set*

$$\hat{E} = \left\{ x \in M, \mid -\frac{1}{2} \leq \mu_N(x) \leq -\frac{1}{n} \right\},$$

where $n > N$. Then

$$A_g(\hat{E}) = 0$$

Proof. We argue by contradiction and assume that

$$A_g(\hat{E}) > 0.$$

Denote $E = M \setminus \hat{E}$ and Σ the set of Lebesgue points of E . For each $x \in \Sigma$ denote by B_x the disk centered at x such that

$$\frac{A_g(E \cap B_x)}{A_g(B_x)} < \epsilon.$$

Let $G = G_x \subset B_x$ be the set defined in Lemma 3.7. Define in $B(x, r)$ the quantity $g' = qg$,

$$f_x(y) = q \frac{\chi(G_x)(y)}{A_{g'}(G_x)}$$

where $\chi(A)$ is the characteristic function of the set A . We introduce the following integral operator T :

$$(14) \quad \begin{aligned} T : L^1(M, g) &\mapsto L^1(M, g) \\ T(h) &= \int_{\Sigma} h(x) f_x dA_g + \tilde{h} \end{aligned}$$

where

$$\tilde{h} = \begin{cases} 0 & \text{on } \Sigma, \\ h & \text{on } M \setminus \Sigma. \end{cases}$$

The operator T preserves the L^1 norm of positive functions on M , i.e. for all $h \in L^1(M)$

$$\int_M T(h)(y) dA_g = \int_M h(y) dA_g.$$

Consider $h \in L^1(M)$, such that $h \geq 0$, $\int_M h = 1$. As a consequence for any $n \geq 1$, we have

$$\int_M T^n(h) dA_g = 1.$$

Set

$$h = \chi(\Sigma)/A_g(\Sigma).$$

Then the sequence $\{T^n(h) dA_g\}_n$ is a sequence of probability measures which contains a subsequence of measures weakly converging to a measure h^* .

Let u be a solution of (5) with $\mu = \mu_N/2$. Then the function $v = u^2$ satisfies the inequality (13) in $B(x, r)$. Hence for $x \in \Sigma$ we have the inequality by Lemma 3.7,

$$u^2(x) < \int u^2(y) f_x(y) dA_g.$$

Thus it follows that the measure h^* is supported on $M \setminus \Sigma$ and

$$\int u^2(y) h(y) dA_g < \int u^2(y) h^*(y) dA_g.$$

Since solutions of (5) are uniformly continuous functions, it follows that we can approximate the measure h^* by a function $s \in L^\infty$ such that $s \geq 0$, has support on $M \setminus \Sigma$,

$$\int s dA_g = 1$$

and

$$\int u^2(y)h(y)dA_g < \int u^2(y)s(y)dA_g.$$

Denote

$$K = \text{ess sup } s + \text{ess sup } h,$$

$$p(x) = (h(x) - s(x))/2Kq(x).$$

Then we have

$$\int u^2 p dA_g < 0.$$

Setting $\bar{\mu}_N = \mu_N + (A_g(G))p$, we have for N large enough

$$-\frac{1}{2} < \bar{\mu}_N < N$$

and the measure $\bar{\mu}_N$ is admissible. On the other hand, we have

$$\int_M v^2 \bar{\mu}_N < \int_M v^2 \mu_N$$

which increases $\tilde{\Lambda}_N$. □

Lemma 3.10. *For any N , we have*

$$A_g(E_-^N) = 0.$$

Proof. . Consider the sequence of sets

$$E_n^N = \left\{ x \in M, \mid -\frac{1}{2} \leq \mu^N(x) \leq -\frac{1}{n} \right\}$$

for $n \geq 1$. By Lemma 3.5 $A_g(E_n^N) = 0$ for all $n \geq 1$. We clearly have

$$E_-^N \subset \bigcup_n E_n^N.$$

This gives the desired result. □

3.2. Control of the eigenfunctions. We start with the following general lemma.

Lemma 3.11. *Let $E \subset M$ be a domain in (M, \tilde{g}) . Let Q be a convex cone in $L^2(M)$ such that if $v \in Q$ then $v \geq 0$. Assume that for all $\varphi \in L^2(M)$ such that $\int_M \varphi = 0$ and $\varphi \geq 0$ on E , there exists $q \in Q$ such that $\int_M \varphi q \geq 0$.*

Then there exists $\tilde{q} \in Q$ such that

- (1) $\tilde{q} \equiv 1$ on $M \setminus E$
- (2) $\int_M \tilde{q} \leq 1$.

Proof. Denote by \mathcal{E} the convex set

$$\mathcal{E} = 1^\perp \bigcap Q$$

where 1^\perp denotes the hyperplane $\{u \in L^2(M) \mid \int_M u = 0\}$. Denote by K the convex cone

$$K = \left\{ u \in Q \mid u \equiv 0 \text{ on } E, \int_M u \leq 0 \right\}.$$

The claim of the theorem amounts to prove that

$$(K + 1) \bigcap \mathcal{E} \neq \emptyset.$$

Assume that this is not the case, i.e. $(K + 1) \bigcap \mathcal{E} = \emptyset$. Since \mathcal{E} and $K + 1$ are two closed convex sets in $L^2(M)$, by Hahn-Banach theorem, there exists a hyperplane \mathcal{H} separating \mathcal{E} from $K + 1$. Let n be a normal vector to the hyperplane \mathcal{H} . We claim that n satisfies the three following properties

- $\int_M n = 0$.
- $n \geq 0$ on E .
- For all $q \in Q$, we have $\int_M qn < 0$.

Therefore, it contradicts the assumptions of the theorem, hence the result. The first point of the claim comes from the construction of n . For the second and third points of the claim, it suffices to notice that, from standard convex analysis, n belongs to the polar cone of $K + 1$, i.e.

$$(K + 1)^* = \left\{ u \in L^2(M) \mid u \geq 0 \text{ on } E, \int_M uq < 0, \forall q \in Q \right\}.$$

□

In our context, the previous lemma admits the following corollary.

Corollary 3.12. *Denote $\bar{g}_N = \mu_N g$. Let $U_1(\bar{g}_N)$ be the eigenspace associated to λ_1 , i.e. the set of functions satisfying*

$$-\Delta_g u = \lambda_1 \mu_N u.$$

Then there exists an orthogonal family $\{u_1^N, \dots, u_\ell^N\} \subset U_1(\bar{g}_N)$ such that if we denote $w = \sum_{i=1}^\ell (u_i^N)^2$ then

- (1) $w \equiv 1$ on $M \setminus E_N$
- (2) $\int_M w \mu_N dA_g \leq 1$

where $E_N = \{x \in M \mid \mu_N(x) = N\}$.

Proof. First notice that $A_{\bar{g}_N}(M) = 1$. We denote

$$\hat{Q} = \{u^2; u \in U_1(\bar{g}_N)\}.$$

Let Q be the convex enveloppe of the cone \hat{Q} .

To be able to apply the previous lemma, we just need to check that for all $\varphi \in L^2(M)$ such that $\int_M \varphi = 0$ and $\varphi \geq 0$ on E_N , there exists $q \in \hat{Q}$ such that $\int_M \varphi q \geq 0$.

Assume the contrary, i.e. there exists $\tilde{\varphi} \in L^2(M)$ such that $\int_M \tilde{\varphi} = 0$, $\tilde{\varphi} \leq 0$ on E_N and for all $q \in \tilde{Q}$, $\int_M \tilde{\varphi} q < 0$. We perturb the potential μ_N by $\tilde{\varphi}$ and denote

$$\tilde{\mu}_N = \mu_N + \varepsilon \tilde{\varphi}.$$

Therefore, on E_N , since $\tilde{\varphi} \leq 0$ we have

$$\tilde{\mu}_N = N + \varepsilon \tilde{\varphi} \leq N$$

and $\tilde{\mu}_N$ is an admissible potential. We claim that if ε is small enough, we have that

$$R_{M, \tilde{\mu}_N g}(u_i) > R_{M, \mu_N g}(u_i),$$

hence a contradiction with the extremality of μ_N . Indeed, we have that

$$\begin{aligned} R_{M, \tilde{\mu}_N g}(u_i) - R_{M, \mu_N g}(u_i) &= \frac{\int_M |\nabla u_i|^2}{\int_M \tilde{\mu}_N u_i^2 \int_M \mu_N u_i^2} \int_M u_i^2 (\mu_N - \tilde{\mu}_N) = \\ &= \frac{\int_M |\nabla u_i|^2}{\int_M \tilde{\mu}_N u_i^2 \int_M \mu_N u_i^2} (-\varepsilon \int_M u_i^2 \tilde{\varphi}). \end{aligned}$$

By assumption on $\tilde{\varphi}$, we have that

$$-\varepsilon \lambda_1 \int_M \tilde{\varphi} u_i^2 > 0,$$

a contradiction. □

4. REGULARITY *a priori* OF LIMITING DENSITIES

We prove here some *a priori* regularity for the limiting density μ_N previously introduced. We introduce the following definition.

Definition 4.1. Let $G \subset F$ be two open subsets of M . Then we denote $Cap_F(G)$ de capacity of G with respect to F as follows:

$$Cap_F(G) = \inf \left\{ \int_F |\nabla u|^2, u \in C_0^\infty(F), u = 1 \text{ on } G \right\}$$

Note that the capacity is of course a subadditive function on the sets G and can be define alternatively as following:

Denote by u_f is a solution of the Dirichlet problem, $\Delta u_f = -f$ in F , $u = 0$ on ∂F . Then

Definition 4.2.

$$Cap_F(G) = \inf \left\{ \int_G f, f \in L_1(G), f \geq 0, u_f \geq 1 \text{ in } G \right\},$$

The following result will be useful.

Lemma 4.3. *Let $u \in H^1(D_1)$ and $\|u\|_{H^1(D_1)} = K$, where D_1 is the unit disk on the plane. Then for every $K, \varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\|u\|_{L^2(D_1)} < \delta.$$

then set

$$\mathcal{E} = \{x \in D_{1/2} \mid |u| > 1\}$$

satisfies

$$Cap_{D_1}(\mathcal{E}) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$.

There exists a partition of unity $\{\varphi_i\}_{i \in I}$ of D_1 such that φ are smooth, $0 \leq \varphi \leq 1$,

$$\sum_{i \in I} \varphi_i \equiv 1$$

and the diameter of every $\text{Supp}(\varphi_i)$ is less than $R > 0$ fixed arbitrary. Furthermore, for every point $z \in D_1$, there are at most 5 (say) functions of $\{\varphi_i\}_{i \in I}$ such that $\varphi_i(z) \neq 0$ and for each $i \in I$

$$\|\varphi_i\|_{H^1} \leq C,$$

where constant C , since the Dirichlet integral is independent under dilations, can be choosen independent on R . We assume without loss of generality that K is sufficiently large such that $K > C$. There exists $\delta_0 = \delta_0(R) > 0$, such that for every $\delta < \delta_0$, we have

$$\sum_{i \in I} \|u \varphi_i\|_{H^1(D_1)} \leq 2K.$$

Let P_y be the Green function with pole at y of the Laplace operator on D_1 . Since

$$-P_y(x) > \frac{1}{2\pi} \ln(1/|x - y|) - 1$$

we have for small $R > 0$,

$$-P_y(x) > \ln(1/R)$$

in D_R . Consider a finite family

$$\{\varphi_i\}_{i=1,\dots,m} \subset \{\varphi_i\}_{i \in I}$$

for which the support of each function has intersection with $D_{1/2}$. We can assume that for $i = 1, \dots, m$

$$\text{Supp}(\varphi_i) \subset D_R^i$$

where D_R^i is a collection of disks of radius R . Denote

$$E_i = \{x \in D_{1/2} \mid u\varphi_i \geq 1/5\}.$$

Therefore, we have

$$\mathcal{E} \subset \bigcup_{i=1}^m E_i$$

We want to estimate the capacity of the sets E_i . Notice first that

$$\text{Cap}_{D_R^i}(E_i) \leq 25 \|u\varphi_i\|_{H^1(\Omega_2)}.$$

Let $y \in D_R^i$ and denote P_y^i the Green function with pole at y of the Laplace operator on D_R^i . Therefore, we have for all $x \in D_R^i$

$$-P_y^i(x) > -P_y^i(x) + \ln(1/R)$$

in D_R^i .

Therefore by definition 4.2, we have

$$\text{Cap}_{D_1}(E_i) \leq \text{Cap}_{D_R^i}(E_i) / \ln(1/R).$$

By subadditivity of the capacity

$$\text{Cap}_{D_1}(\mathcal{E}) \leq \sum_{i=1}^m \text{Cap}_{D_1}(E_i)$$

and then

$$\text{Cap}_{D_1}(\mathcal{E}) \leq \frac{K}{\ln(1/R)}$$

and the result follows by choosing R small enough and correspondingly small $\delta_0(R)$, such that

$$\text{Cap}_{D_1}(\mathcal{E}) \leq \varepsilon.$$

□

First we exclude that the limiting density blows up to a point. The following result was proved by A. Girouard in [Gir09] but for the convenience of the reader we give some details. Let μ be the weak limit of μ_N on (M, g) .

Theorem 4.4. *Assume that $\tilde{\Lambda}(M) > 8\pi$. Then the measure μdA_g is not a Dirac measure.*

We argue by contradiction and assume that

$$\mu_N \rightharpoonup^* \delta_{x_0} \text{ weakly in } \mathcal{M}(M)$$

for some $x_0 \in M$, as $N \rightarrow +\infty$.

First, we denote $\bar{g}_N = \mu_N g$, a sequence of metrics on M .

We split the proof into several lemmata. Let $D = D(x_0, r) \subset M$ be a disk of radius $r > 0$ small enough, with center x_0 . Consider now the restriction of the metric \bar{g}_N to D denoted $\bar{g}_{N,D}$. Consider now a conformal map f^{-1} from D into the sphere. Such a map exists since D is a subdomain of a Riemannian surface. We denote \tilde{g}_N , the pull-back of $\bar{g}_{N,D}$ via f , i.e.

$$\tilde{g}_N = f^* \bar{g}_{N,D},$$

which is a metric on \mathbb{S}^2 .

We introduce now the Moebius map. More precisely, there exists a unique vector $e \in \mathbb{R}^3$, $|e| < 1$ such that the map

$$(15) \quad \begin{aligned} \sigma_e : \mathbb{S}^2 &\mapsto \mathbb{S}^2 \\ \sigma_e(x) &= \frac{(1-|e|^2)x - (1-2e \cdot x + |x|^2)e}{1-2e \cdot x + |e|^2|x|^2} \end{aligned}$$

is conformal and satisfies the orthogonality conditions for $i = 1, 2, 3$

$$\int_{\mathbb{S}^2} x_i \circ \sigma_e dA_h = 0,$$

where x_i are coordinate functions in \mathbb{R}^3 . One can find this result, often referred as Hersch result, in [Her70] and [SY94].

Consider now the following metric on \mathbb{S}^2

$$\hat{g}_N = \sigma_e^* \tilde{g}_N,$$

for which the following orthogonality conditions hold

$$\int_{\mathbb{S}^2} x_i dA_{\hat{g}_N} = 0.$$

One can assume without loss of generality that the metric \hat{g}_N is conformally equivalent to the standard metric and then

$$R_{\mathbb{S}^2, \hat{g}_N}(x_i) \leq 8\pi = \Lambda_{g_{\text{round}}}(\mathbb{S}^2).$$

The following lemma shows that the Moebius map allows to shrink the complement of the image under f of the singularity to a point.

Lemma 4.5. *There exists $x_1(\in \mathbb{S}^2) \neq f^{-1}(x_0)$ such that*

$$\lim_{N \rightarrow +\infty} \text{dist}_{\hat{g}_N}(\mathbb{S}^2 \setminus f^{-1}(D), x_1) = 0,$$

where $\text{dist}_{\hat{g}_N}$ is the Hausdorff distance between sets induced by the metric on $(\mathbb{S}^2, \hat{g}_N)$.

Proof. This follows from the explicit expression of σ_e , the orthogonality conditions and the fact that μ_N concentrates on x_0 . \square

Lemma 4.6. *For all $\delta > 0$, there exists a smooth function ψ supported in $f^{-1}(D)$ such that*

$$R_{\mathbb{S}^2, \hat{g}_N}(\psi) \leq 4\pi A_{\hat{g}_N}(\mathbb{S}^2) + \delta$$

for all $\delta > 0$.

Proof. Fix $\varepsilon > 0$. Let φ be a capacity function on the sphere \mathbb{S}^2 with support on a disk of small radius around x_1 , such that

$$\int_{\mathbb{S}^2} |\nabla \varphi|^2 dA_{\hat{g}_N} < \varepsilon.$$

Consider now the optimal function $\tilde{\varphi}_N$ for the Rayleigh quotient $R_{\mathbb{S}^2, \hat{g}_N}$. By one of Hersch's results (see [Her70]), we have

$$R_{\mathbb{S}^2, \hat{g}_N}(\tilde{\varphi}_N) \leq 4\pi A_{\hat{g}_N}(\mathbb{S}^2).$$

Consider now the test function

$$\psi = (\tilde{\varphi}_N - \bar{\varepsilon})(1 - \varphi),$$

where $\bar{\varepsilon}$ is such that $\int_{\mathbb{S}^2} \psi dA_{\hat{g}_N} = 0$. Therefore, by the previous lemma: for all $\delta > 0$,

$$R_{\mathbb{S}^2, \hat{g}_N}(\psi) \leq R_{\mathbb{S}^2, \hat{g}_N}(\tilde{\varphi}_N) - \delta,$$

hence the desired result. \square

Lifting the function ψ on M we get a contradiction .

Lemma 4.7. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\text{cap}(E) < \delta$ then $\int_E \mu dA_g < \varepsilon$ for any $E \subset M$.*

Proof. If there were not true, by Theorem 4.4, there are two closed subsets $E_1 \subset E \subset M$ and $E_2 \subset\subset E \subset M$ and $\varepsilon > 0$ such that for all $\delta > 0$

$$E_1 \cap E_2 = \emptyset, \quad \int_{E_1} \mu dA_g, \int_{E_2} \mu dA_g \geq \varepsilon$$

and $\text{cap}_M(E_1) < \delta$, $\text{cap}_M(E_2) < \delta$. Denoting \tilde{E}_1, \tilde{E}_2 , two $\text{dist}(E_1, E_2)/2$ -neighborhoods of E_1 and E_2 , we have by the definition of the capacity: there exists functions $f_i \in C_0^\infty(\tilde{E}_i)$ such that $f_i \geq 1$ on E_i and

$\int_M |\nabla f_i|^2 \leq \delta$. On the other hand, by the weak convergence of the sequence μ_N , we have for $i = 1, 2$

$$\int_M f_i^2 \mu_N dA_g \rightarrow \int_M f_i^2 \mu dA_g.$$

This gives that

$$\lim_{N \rightarrow +\infty} \lambda_1(\mu_N g) = 0,$$

a contradiction. \square

The previous lemma has the following corollaries.

Corollary 4.8. *Let $v \in H^1(M) \cap L^\infty(M)$. Then $v\mu$ is a Radon measure on M .*

Corollary 4.9. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $v \in H^1(M)$, $|v| < 1$ and $\|v\|_{H^1} < \delta$, then*

$$\int_M |v\mu| dA_g < \varepsilon.$$

5. PROOFS OF THEOREMS 2.1 AND 2.2

We now reach the conclusions of our Theorems 2.1 and 2.2, since we have all the ingredients to control the weak limit of the sequence μ_N . We have up to extraction of subsequences

$$\begin{aligned} u_i^N &\rightharpoonup u_i, \text{ weakly in } H^1(M), i = 1, \dots, \ell, \\ \mu^N &\rightharpoonup^* \mu, \text{ weakly in } \mathcal{M}_g(M), \\ \tilde{\Lambda}_N &\rightarrow \tilde{\Lambda} \end{aligned}$$

Furthermore, the limiting density μ satisfies

$$\mu > 0 \text{ on a.e. in } M$$

The last statement follows from the continuity in weak topology of the first eigenvalue with respect to the metric. The next theorem is a direct consequence of Lemma 4.3.

Theorem 5.1. *Let $\{u_n\}_{n \geq 0}$ be a sequence in $H^1(M)$ weakly converging to a function $u \in H^1(M)$. Then for all $\varepsilon > 0$ there exists a set $\Omega \subset M$ such that*

- $\text{Cap}_M(\Omega) < \varepsilon$.
- There exists a subsequence $\{u_{n_k}\}_{k \geq 0}$ such that

$$u_{n_k} \rightarrow u$$

uniformly on $M \setminus \Omega$ as $k \rightarrow +\infty$.

Proof. Consider the sequence of functions $v_n := u_n - u$. We have v_n bounded in $H^1(M)$ and weakly converging to 0 in $H^1(M)$. By Sobolev embedding, up to an extraction, the sequence v_n converges strongly in $L^2(M)$. We introduce for fixed $N > 0$

$$\Omega_n^N = \left\{ x \in M \mid |v_n| > \frac{1}{N} \right\}.$$

By Lemma 4.3, we have that $\text{Cap}(\Omega_n^N)$ is arbitrary small as $n \rightarrow +\infty$ and the conclusion follows. \square

The next lemma ensures that the functions u_i are eigenfunctions of the extremal $\tilde{\Lambda}_1$.

Lemma 5.1. *The function u_i are eigenfunctions in a weak sense: for all $\varphi \in H^1(M) \cap L_\infty(M)$ and $i = 1, \dots, \ell$ we have*

$$\int_M \nabla u_i \cdot \nabla \varphi \, dA_g = \tilde{\Lambda}_1 \int_M \mu u_i \varphi \, dA_g.$$

Proof. By definition, we have for all $\varphi \in H^1(M)$, $i = 1, \dots, \ell$ and any $N > 1$

$$\int_M \nabla u_i^N \cdot \nabla \varphi \, dA_g = \tilde{\Lambda}_N \int_M \mu_N u_i^N \varphi \, dA_g.$$

By the weak H^1 convergence of the sequence $\{u_i^N\}_N$, we have

$$\int_M \nabla u_i^N \cdot \nabla \varphi \, dA_g \rightarrow \int_M \nabla u_i \cdot \nabla \varphi \, dA_g.$$

Denote,

$$\lim_{N \rightarrow +\infty} \tilde{\Lambda}_N = \tilde{\Lambda}_1.$$

By Theorem 5.1, there exist sets $\Omega_k \subset M$, $k = 1, 2, \dots$ such that for all $i = 1, \dots, \ell$, u_i^N converges uniformly to a function u_i on $M \setminus \Omega_k$ and furthermore $\text{cap}_M(\Omega_k) < 1/k$. We then can define functions $b_k \in H^1(M)$, $0 \leq b_k \leq 1$, $b_k = 0$ on Ω_k and $\|1 - b_k\|_{H^1(M)} \rightarrow 0$ as $k \rightarrow \infty$. Since u_i^N converges uniformly to a function u_i on $M \setminus \Omega_k$ we can pass to the limit in the integral identity

$$\int_M \nabla u_i^N \cdot \nabla (b_k \varphi) \, dA_g = \tilde{\Lambda}_N \int_M \mu_N u_i^N (b_k \varphi) \, dA_g,$$

as $N \rightarrow \infty$ and get

$$\int_M \nabla u_i \cdot \nabla (b_k \varphi) \, dA_g = \tilde{\Lambda}_1 \int_M \mu u_i (b_k \varphi) \, dA_g.$$

It follows that

$$|u_i| \leq 1.$$

Hence from Corollary 4.9 it follows that we can pass to the limit in the last identity as $k \rightarrow \infty$ and get the desired result. \square

As a corollary of Lemma 5.1 we have

Corollary 5.2. *The following equality holds in the sense of distributions:*

$$-\Delta u_i = \tilde{\Lambda}_1 \mu u_i \quad i = 1, \dots, \ell.$$

It just a bit more than a formal computation to get the following lemma

Lemma 5.3. *The following equality holds in the sense of distributions:*

$$\Delta u_i^2 = 2|\nabla u_i|^2 - \tilde{\Lambda}_1 \mu u_i^2 \quad i = 1, \dots, \ell.$$

Proof. It will be sufficiently to prove that the identity holds on any small subdomain of M . Thus without loss we may assume that the function u_i defined in a disk on the plane and Δ is the Laplacian on the plane.

Let $a(x) \in C(\mathbb{R}^2)$, $a \geq 0$, a supported by a unit disk and

$$\int a dx = 1.$$

Set $a_t(x) = \frac{1}{t^2} a(\frac{x}{t})$ (such that its integral is 1), $u_t = u_i * a_t$. Then $u_t \rightarrow u_i$ in H^1 , $|\nabla u_t|^2 \rightarrow |\nabla u_i|^2$ in L_1 and weakly, $\Delta u_t^2 \rightarrow \Delta u_i^2$ weakly, $a_t * \mu u_i \rightarrow \mu u_i$ weakly. Thus $u_t(a_t * \mu u_i) \rightarrow \mu u_i^2$ weakly. Since

$$\Delta u_t^2 = 2|\nabla u_t|^2 + 2u_t \Delta u_t = 2|\nabla u_t|^2 - 2u_t(a_t * \Delta u_i) = 2|\nabla u_t|^2 - 2u_t(a_t * \mu u_i),$$

passing to the limit as $t \rightarrow 0$ we finish the proof. \square

From Lemma 3.2 and Corollary 3.12 it follows that

$$\sum u_i^2 = 1,$$

a.e. in M , i.e., we have the following result

Lemma 5.4. *The map $\phi = (u_1, \dots, u_\ell) : M \rightarrow \mathbb{S}^{\ell-1}$ is well defined a.e. on M .*

Applying the Laplace-Beltrami Δ to the last identity and by Lemma 5.3, it follows that

$$\sum \tilde{\Lambda}_1 \mu u_i^2 - \sum |\nabla u_i|^2 = 0.$$

Thus

$$(16) \quad \mu = \sum |\nabla u_i|^2 / \tilde{\Lambda}_1.$$

The last equality implies that the limit measure μ has an L^1 density and moreover the map ϕ is a weak solution of the harmonic map equation, namely

$$\sum u_i \Delta u_i = \sum |\nabla u_i|^2.$$

Hence by the result of Helein [Hél90], ϕ is a smooth harmonic map of M into the sphere. We give here an alternative proof of the last result.

From Lemma 5.1 we have

$$\int_M \mu u_i \varphi dA_g = 0.$$

Since $u_i \in H^1(M) \cap L_\infty(M)$, $\mu \in L_1(M)$ it follows that

$$\tilde{\Lambda}_1 = \inf_{u \in E} R(u),$$

where

$$R(u) = \frac{\int_M |\nabla u|^2 dA_g}{\int_M u^2 \mu dA_g}$$

and the infimum is taken over the space

$$E = \left\{ u \in H^1(M) \cap L_\infty, \quad \int_M u \mu dA_g = 0 \right\}.$$

Set

$$u^+ = \sup\{0, u\}.$$

Again from Lemma 5.1 we conclude

$$R(u) = R(u^+).$$

Lemma 5.5. *The map $\phi = (u_1, \dots, u_\ell)$ previously defined is harmonic from M into $\mathbb{S}^{\ell-1}$, i.e. ϕ minimizes in $H^1(M, \mathbb{S}^{\ell-1})$ the Dirichlet form*

$$\mathcal{D}(\psi) = \int_M |D\psi|^2 dA_{\bar{g}}$$

Proof. Suppose that the map ϕ is not harmonic. Therefore, for all $\varepsilon > 0$, there exist $E \subset M$ such that $\text{diam}(E) < \varepsilon$ and a map $\psi : M \mapsto \mathbb{S}^{\ell-1}$ such that

$$\int_E |D\psi|^2 \mu dA_g < \int_E |D\phi|^2 \mu dA_g$$

and

$$\psi = \phi \quad \text{on } M \setminus E.$$

We choose coordinates on the sphere $\mathbb{S}^{\ell-1}$ such that $\psi(E)$ is in the positive octant. In these coordinates, we still have on E

$$\sum_{i=1}^{\ell} \psi_i^2 \equiv 1$$

and then

$$\int_E \sum_{i=1}^{\ell} \psi_i^2 \mu dA_g = \int_E \sum_{i=1}^{\ell} u_i^2 \mu dA_g.$$

Then there is component ψ_k such that

$$(17) \quad R(\psi_k) < R(u_k).$$

Set

$$u = \frac{\int_M u_k^+ \mu dA_g}{\int_M \psi_k^+ \mu dA_g} \psi_k^+ - (-u_k)^+.$$

Then $u \in E$ and from 17

$$R(u) < R(u_k),$$

a contradiction. □

We can conclude the proofs of our results. From Morrey's regularity result [Mor66], it follows that all the eigenfunctions u_i are real analytic. Hence the density μ is a real analytic function on M , positive outside an analytic manifold γ . Since μ is not identically zero it follows that the dimension of γ is either 0 or 1. Assume that the dimension of γ is 1. Then from formula 16 it follows that $|\nabla u_i| = 0$ on γ . Let γ' be a connected component of γ . Then it follows that all u_i are constants on γ' . Assume without loss that $u_1 = 0$ on γ' . Thus $u_1 = |\nabla u_1| = 0$ on γ' and by the uniqueness of the solution of Cauchy problem it follows that u_1 is identically zero. Hence the dimension of γ is 0 and thus γ is a set of at most a finite number of points on M , and the theorems are proved.

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